# 4.1.3 Fixpoint Semantics of Logic Programs

Idea: Define the semantics of programs in a mathematical way as the fixpoint of some function (denotational semantics).

Fixpoint of a fct. $f(x) = x$

E.g. fixpoints of square: $\mathbb{N} \to \mathbb{N}$ are $0, 1$.

Solution: Function $trans_{\mathcal{P}}$ transforms a set of atomic formulas without variables into a new set of atomic formulas without variables.

$trans_{\mathcal{P}}(M)$ should contain $M$ and all statements that can be derived from $M$ in one step (with the program $\mathcal{P}$).

Thus: one only looks at the program $\mathcal{P}$ and computes all true statements about $\mathcal{P}$.

$\emptyset$ all true statements that can be proved in $0$ steps

$trans_{\mathcal{P}}(\emptyset)$ all true statements that can be proved in at most 1 step (i.e., all ground instances of all facts in $\mathcal{P}$).

$trans_{\mathcal{P}}(trans_{\mathcal{P}}(\emptyset))$ all true statements that can be proved in at most 2 steps

$\vdots$

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Def 4.1.9. ($trans_{\mathcal{P}}$)

Let $\mathcal{P}$ be a LP over a signature $(\Sigma, \Delta)$.

Then $trans_{\mathcal{P}} : \underbrace{Pot(At(\Sigma, \Delta, \emptyset))}_{\text{set of all sets of ground atomic formulas}} \to Pot(At(\Sigma, \Delta, \emptyset))$

is defined as follows:

$$trans_{\mathcal{P}}(M) = M \cup \{A' \mid \{A', \neg B_1', \dots, \neg B_n'\} \text{ is a ground instance of a clause } \{A, \neg B_1, \dots, \neg B_n\} \in \mathcal{P}, \text{ and } B_1', \dots, B_n' \in M\}$$

Then we define:

$$M_{\mathcal{P}} = \emptyset \cup trans_{\mathcal{P}}(\emptyset) \cup trans_{\mathcal{P}}^2(\emptyset) \cup \ldots = \bigcup trans_{\mathcal{P}}^i(\emptyset)$$

Idea: $M_{\mathcal{P}}$ contains all true ground statements about the prog. $\mathcal{P}$. Therefore, one can use $M_{\mathcal{P}}$ to define the semantics of $\mathcal{P}$ w.r.t. a query $G$ (one only has to check which ground instances of $G$ are contained in $M_{\mathcal{P}}$).

Ex. 4.1.10 We regard the prog $\mathcal{P}$ from Ex. 4.1.2.

$trans_{\mathcal{P}}^0(\emptyset) = \emptyset$

$trans_{\mathcal{P}}(\emptyset) = \{motherOf(ren, sus), married(gerd, ren)\}$

$trans_{\mathcal{P}}^2(\emptyset) = trans_{\mathcal{P}}(\emptyset) \cup \{fatherOf(gerd, sus)\} = M_{\mathcal{P}}$

$trans_{\mathcal{P}}^i(\emptyset) = trans_{\mathcal{P}}^2(\emptyset)$ for all $i \geq 2$.

Thus: $M_{\mathcal{P}}$ is reached after 2 iterations.

Ex. 4.1.11 In general, the computation of $M_{\mathcal{P}}$ can require infinitely many iterations.

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p(a).
p(f(X)) :- p(X).
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Thus, $\mathcal{P} = \{ \ \{p(a)\},$

$$\{p(f(X)), \neg p(X)\} \quad \}$$

$$trans_{\mathcal{P}}^{0}(\emptyset) = \emptyset$$
$$trans_{\mathcal{P}}(\emptyset) = \{p(a)\}$$
$$trans_{\mathcal{P}}^{2}(\emptyset) = \{p(a), p(f(a))\}$$
$$trans_{\mathcal{P}}^{3}(\emptyset) = \{p(a), p(f(a)), p(f(f(a)))\}$$
$$\vdots$$

$$M_{\mathcal{P}} = \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^{i}(\emptyset) = \{p(f^{i}(a)) \mid i \in \mathbb{N}\}$$

This construction computes a fixpoint of the function $trans_{\mathcal{P}}$, i.e., $trans_{\mathcal{P}}(M_{\mathcal{P}}) = M_{\mathcal{P}}$

Thus: $M_{\mathcal{P}}$ contains all true statements about $\mathcal{P}$, it does not have to be extended further

Moreover, $M_{\mathcal{P}}$ is the least fixpoint of $trans_{\mathcal{P}}$ (i.e., it is a subset of all other fixpoints of $trans_{\mathcal{P}}$)

Thus: $M_{\mathcal{P}}$ only contains those statements about $\mathcal{P}$ that are necessary, i.e., it does not contain more than the true statements about $\mathcal{P}$.

The claim that $M_{\mathcal{P}}$ is the least fixpoint of $trans_{\mathcal{P}}$ can be proved formally.

Then: Define the semantics of $\mathcal{P}$ via the least fixpoint of $trans_{\mathcal{P}}$.

# A. Properties of $\subseteq$ (subset relation)

- reflexive: $M_1 \subseteq M_1$
- antisymmetric: $M_1 \subseteq M_2$ and $M_2 \subseteq M_1$ implies $M_1 = M_2$
- transitive: $M_1 \subseteq M_2$ and $M_2 \subseteq M_3$ implies $M_1 \subseteq M_3$

ordering

Thus: $\subseteq$ is a reflexive ordering

Moreover, $\subseteq$ is "complete":

$\emptyset$, $trans_\rho(\emptyset)$, $trans_\rho^2(\emptyset)$, ....

is a sequence of sets with

$\emptyset \subseteq trans_\rho(\emptyset) \subseteq trans_\rho^2(\emptyset) \subseteq \ldots$

Such sequences are called <u>chains</u>.

A reflexive ordering is complete iff
- it has a smallest element
- every chain has a least upper bound

<u>Lemma 4.1.12</u> (Completeness of $\subseteq$)

The relation $\subseteq$ is complete on $Pot(At(\Sigma, \Delta, \mathcal{V}))$:
The smallest set is $\emptyset$ and every chain

$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ (finite or infinite)

has the least upper bound $M = \bigcup_{i \in \mathbb{N}} M_i$.

Proof: $\emptyset$ is the smallest element, because $\emptyset \subseteq M'$ for all $M' \subseteq At(\Sigma, \Delta, \emptyset)$.

$M$ is an upper bound of the chain, because $M_i \subseteq M$ for all $i \in \mathbb{N}$.

To show that $M$ is the least upper bound of $M_0, M_1, \ldots$, assume that there is another upper bound $M'$ with $M_0 \subseteq M'$, $M_1 \subseteq M'$, ....

This implies $M = \bigcup_{i \in \mathbb{N}} M_i \subseteq M'$ $\boxed{\phantom{x}}$

## Fixpoint Semantics

this is a chain

① $trans_{\mathcal{P}}^0(\emptyset) = \emptyset$ $\quad trans_{\mathcal{P}}^0(\emptyset) \subseteq$

$trans_{\mathcal{P}}(\emptyset)$ $\quad trans_{\mathcal{P}}(\emptyset) \subseteq$

$trans_{\mathcal{P}}^2(\emptyset)$ $\quad trans_{\mathcal{P}}^2(\emptyset) \subseteq \ldots$

$\vdots$

This is the least upper bound of this chain.

Take $\bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset) = M_{\mathcal{P}}$. This the set of all true statements about $\mathcal{P}$.

② Take the least fixpoint of $trans_{\mathcal{P}}$.

(i.e., the smallest set $M$ with $trans_{\mathcal{P}}(M) = M$).

This is the set of all true statements about $\mathcal{P}$.

We now show that both alternatives ① and ② yield the same set of "true" statements.

This is due to properties of the relation $\subseteq$ and of the function $trans_\rho$.

## B. Properties of the function $trans_\rho$

$trans_\rho$ has 2 important properties:

It is monotonic and continuous.
("stetig")

**Lemma 4.1.13** (Monotonicity and Continuity of $trans_\rho$)

(a) The function $trans_\rho$ is monotonic, i.e.,
if $M_1 \subseteq M_2$ then $trans_\rho(M_1) \subseteq trans_\rho(M_2)$.

(b) The function $trans_\rho$ is continuous, i.e.,
for every chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$
we have $trans_\rho(\bigcup_{i \in \mathbb{N}} M_i) = \bigcup_{i \in \mathbb{N}} trans_\rho(M_i)$.

$$\begin{tikzcd}
M_0 \arrow[r, phantom, "\subseteq"] \arrow[d] & M_1 \arrow[r, phantom, "\subseteq"] \arrow[d] & M_2 \arrow[r, phantom, "\ldots"] \arrow[d] & {} \arrow[r, "lub"] & M \arrow[d] \\
trans_\rho(M_0) \arrow[r, phantom, "\subseteq"] & trans_\rho(M_1) \arrow[r, phantom, "\subseteq"] & trans_\rho(M_2) \arrow[r, phantom, "\ldots"] & {} \arrow[r, "lub"] & trans_\rho(M)
\end{tikzcd}$$

Proof: (a) is straightforward.
We show (b):

$$trans_\rho\left(\bigcup_{i \in \mathbb{N}} M_i\right) \supseteq \bigcup_{i \in \mathbb{N}} trans_\rho(M_i)$$

This follows from monotonicity of $trans_P$.

$M_i \subseteq \bigcup_{i \in \mathbb{N}} M_i$ for all $i$

$\rightsquigarrow trans_P(M_i) \subseteq trans_P(\bigcup_{i \in \mathbb{N}} M_i)$ by monot. of $trans_P$

for all $i$

$\rightsquigarrow \bigcup_{i \in \mathbb{N}} trans_P(M_i) \subseteq trans_P(\bigcup_{i \in \mathbb{N}} M_i)$

$\underline{trans_P(\bigcup_{i \in \mathbb{N}} M_i) \subseteq \bigcup_{i \in \mathbb{N}} trans_P(M_i)}$

Let $A' \in trans_P(\bigcup_{i \in \mathbb{N}} M_i)$.

Then $A' \in \bigcup_{i \in \mathbb{N}} M_i$ or there is a ground instance $\{A', \neg B_1', \dots, \neg B_n'\}$ of a program clause from $P$, where $B_1', \dots, B_n' \in \bigcup_{i \in \mathbb{N}} M_i$.

If $A' \in \bigcup_{i \in \mathbb{N}} M_i$, then there exists an $M_j$ with $A' \in M_j$.

Thus: $A' \in trans_P(M_j) \subseteq \bigcup_{i \in \mathbb{N}} trans_P(M_i)$.

Otherwise, since $M_0 \subseteq M_1 \subseteq \dots$ there exists an $M_j$ with $B_1', \dots, B_n' \in M_j$.

Thus: $A' \in trans_P(M_j) \subseteq \bigcup_{i \in \mathbb{N}} trans_P(M_i)$. ∎

General
<u>Fixpoint Theorem:</u> (Kleene/Tarski)

If $f$ is a continuous function over a complete

reflexive ordering, then $f$ has a least fixpoint.
This least fixpoint is the least upper bound of
the chain $f^0(\perp), f^1(\perp), f^2(\perp), \ldots$

smallest element of the complete ordering

Thm 4.1.14 (Fixpoint Thm)

For every LP $\mathcal{P}$, $trans_{\mathcal{P}}$ has a least fixpoint $lfp(trans_{\mathcal{P}})$.
We have $lfp(trans_{\mathcal{P}}) = \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset)$.

Proof: 1. Prove that $\bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset)$ is a fixpoint of $trans_{\mathcal{P}}$.

$$trans_{\mathcal{P}}\left(\bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset)\right)$$

$$= \bigcup_{i \in \mathbb{N}} \left(trans_{\mathcal{P}}\left(trans_{\mathcal{P}}^i(\emptyset)\right)\right)$$ by continuity of $trans_{\mathcal{P}}$, Lemma 4.1.13 (b)

$$= \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^{i+1}(\emptyset)$$

$$= \emptyset \cup \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^{i+1}(\emptyset)$$

$$= \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset)$$

2. Prove that $\bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^i(\emptyset) \subseteq M$ holds for all fixpoints $M$ of $trans_{\mathcal{P}}$.

We show that $trans_{\mathcal{P}}^i(\emptyset) \subseteq M$ holds for all $i$, by induction on $i$.

<u>Ind. Base</u>: $i=0$ $\quad trans_{\mathcal{P}}^0(\emptyset) = \emptyset \subseteq M$ ✓

<u>Ind. Step</u>: $i>0$

Ind. Hyp: $trans_{\mathcal{P}}^{i-1}(\emptyset) \subseteq M$

$\curvearrowright \underbrace{trans_{\mathcal{P}}(trans_{\mathcal{P}}^{i-1}(\emptyset))}_{trans_{\mathcal{P}}^{i}(\emptyset)} \subseteq \overbrace{trans_{\mathcal{P}}(M)}^{M\text{, since } M \text{ is a fixpoint of } trans_{\mathcal{P}}}$ by monotonicity of $trans_{\mathcal{P}}$ (Lemma 4.1.13 (a))

$\boxed{q.e.d.}$

<u>Def 4.1.15</u> (Fixpoint Semantics of Logic Programming)

Let $\mathcal{P}$ be a LP, let $G = \{\neg A_1, \dots, \neg A_k\}$ be a query. Then the fixpoint semantics of $\mathcal{P}$ w.r.t. the query $G$ is:

$$\mathcal{F}[\![\mathcal{P}, G]\!] = \{\sigma(A_1 \wedge \dots \wedge A_k) \mid \sigma(A_i) \in lfp(trans_{\mathcal{P}}) \text{ for all } 1 \leq i \leq k\}.$$

<u>Thm 4.1.16</u> (Equivalence of Fixpoint Semantics to the other Semantics)

Let $\mathcal{P}$ be a LP, let $G$ be a query.

Then $\mathcal{F}[\![\mathcal{P}, G]\!] = \mathcal{D}[\![\mathcal{P}, G]\!] = \mathcal{P}[\![\mathcal{P}, G]\!]$.

<u>Proof</u>: By Thm 4.1.8 we have $\mathcal{D}[\![\mathcal{P}, G]\!] = \mathcal{P}[\![\mathcal{P}, G]\!]$.

We show $\mathcal{P}[\![\mathcal{P}, G]\!] \subseteq \mathcal{F}[\![\mathcal{P}, G]\!]$ (Course notes

induction on the length of
the SLD-resolution proof)

and $\underline{\mathcal{F}[\![\mathcal{P}, G]\!] \subseteq \mathcal{D}[\![\mathcal{P}, G]\!]}$.

Let $\sigma(A_1 \wedge \ldots \wedge A_k) \in \mathcal{F}[\![\mathcal{P}, G]\!]$.

Thus, $\sigma(A_i) \in lfp(trans_{\mathcal{P}})$ for all $1 \leq i \leq k$.

We have to show $\mathcal{P} \models \sigma(A_i)$.

By the fixpoint theorem, we have

$$\sigma(A_i) \in lfp(trans_{\mathcal{P}}) = \bigcup_{i \in \mathbb{N}} trans_{\mathcal{P}}^{i}(\emptyset)$$

Thus, there is a $j \in \mathbb{N}$ with $\sigma(A_i) \in trans_{\mathcal{P}}^{j}(\emptyset)$.

So it suffices to show the following for all $j \in \mathbb{N}$:

$\underline{A' \in trans_{\mathcal{P}}^{j}(\emptyset) \quad \text{implies} \quad \mathcal{P} \models A'}$

We use induction on $j$.

<u>Ind. Base</u> $j = 0$ $\quad$ $A' \in trans_{\mathcal{P}}^{0}(\emptyset) = \emptyset$ ✓
is impossible

<u>Ind. Step</u> $j > 0$

$A' \in trans_{\mathcal{P}}^{j}(\emptyset) = trans_{\mathcal{P}}(trans_{\mathcal{P}}^{j-1}(\emptyset))$

If $A' \in trans_{\mathcal{P}}^{j-1}(\emptyset)$, then $\mathcal{P} \models A'$ by the ind. hyp.

Otherwise, there is a ground instance
$\{A', \neg B_1', \ldots, \neg B_n'\}$ of a prog. clause in $\mathcal{P}$

with $B_1', \dots, B_n' \in \mathrm{trans}_{\mathcal{P}}^{j-1}(\emptyset)$.

By the ind. hyp, we have $\mathcal{P} \models B_1'$, ..., $\mathcal{P} \models B_n'$.

Moreover, $\mathcal{P} \models A' \vee \neg B_1' \vee \dots \vee B_n'$

Hence, $\mathcal{P} \models A'$. $\square$